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A Weierstrass Theorem for Real Banach Spaces

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Let X be a real separable Banach space and K be a compact subset; $C(K)$ denotes the family of continuous functions from K into X together with the uniform norm topology

$$\| f - g \| = \max_{x \in K} \| f(x) - g(x) \|.$$

In some recent papers [3, 4], Prenter has proved that if X is a Hilbert space or a separable Banach space with "Property M" then the Weierstrass theorem holds in $C(K)$. It is the purpose of the present note to give a general theorem without any restriction on X. Our method is entirely different from Prenter's.

1. PRELIMINARIES

Let X be a Banach space and $k \geqslant 1$. X^k denotes the Banach space

$$\underbrace{X \times \cdots \times X}_{k \text{ times}}.$$

DEFINITION 1.1. A k-linear operator T on X is a function on X^k into X which is linear in each of its arguments separately.

A 0-linear operator L_0 on X is a constant function on X into X and we shall identify a 0-linear operator L_0 with its range.

The norm of a k-linear operator T is the number

$$\| T \| = \sup \| T(x_1, \ldots, x_k) \| / \| x_1 \| \| x_2 \| \cdots \| x_k \| .$$

If T is a k-linear operator and S is a q-linear operator then we can define in a natural way the products TS and ST which are obviously $(k + q)$-linear operators. By a polynomial we mean any function of the form

$$P_n(x) = L_0 x + \cdots + L_n x^n,$$

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where L_i are i-linear operators. The number n is called the degree, $n = \deg P_n$.

Some explanation of notation is in order. For any $k \geqslant 1$ and T a k-linear operator on X we note

$$T(x, x,..., x) = Tx^k.$$

It is obvious that the set of polynomials forms an algebra (generally non-commutative) with properties:

1. $\deg(P_n + P_m) \leqslant \max\{\deg P_n, \deg P_m\}$,
2. $\deg(P_n \cdot P_m) \leqslant n + m$.

For details see any book of functional analysis.

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We consider now a Banach space X; we let K be a compact subset in X and introduce

(1) $C(K, X) = \{f, f: K \to X \text{ continuously}\}$,

(2) $P(K, X) = \{p, p \text{ is a continuous polynomial}\}$,

and if $Y \subset R$ (the reals)

(3) $C(K, Y) = \{f, f: K \to Y \text{ continuously}\}$,

(4) $P(K, Y) = \{f, f \text{ is a continuous polynomial from } K \text{ to } Y\}$. Since all spaces in (1)–(3) are clearly defined, we give some details for (4). By real polynomial we mean any function of the form

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where a_0 is a real constant and

$$a_i: \underbrace{X \times \cdots \times X}_{i \text{ times}} \to Y \qquad (Y \text{ the reals}),$$

with $a_i(x,..., x) = a_i x^i$. It is clear that $P(K, Y)$ forms an algebra. Our first basic result gives a relation between $C(K, Y)$ and $P(K, Y)$.

THEOREM 2.1. *The algebra $P(K, R)$ is dense in the uniform norm topology in $C(K, R)$.*

Proof. Our proof depends upon a basic idea used by de Branges [1]. This consists of remarking that the support of some measures has special properties with respect to some subspaces and using this for commutative subalgebras of the algebra of all continuous functions on a compact Hausdorff space.

We need the following notion.

DEFINITION 2.1. A level set for a family of functions $S \subset C(K, R)$ is a subset $K_0 \subset K$ such that S/K_0 contains only constant functions. It is obvious that any point of K is a level set, and using Zorn's lemma we see that any point is contained in a maximal level set; the same is true for any level set.

LEMMA 2.2. *Let F be a continuous linear functional on $C(K, R)$ such that*

(1) *F is an extreme point of the set of all linear functionals λ on $C(K, R)$ such that $\| \lambda \| \leqslant 1$ and λ annihilates $P(K, R)$,*

(2) *μ is the measure determining F so that $F(f) = \int_K f(s)\, d\mu(s)$. Then the support of μ,* sup μ, is a level set of $P(K, R)$.

Proof. Suppose that is not so. We find an element of $P(K, R)$, say g_0, such that it is not constant on sup μ (which is equal to sup $\| \mu \|$, $\| \mu \|$ the total variation of μ). We may suppose without loss of generality that

$$0 \leqslant g_0 \leqslant 1.$$

We can consider also the following measures

$$\mu_1 = g_0 \mu, \qquad \mu_2 = (1 - g_0)\, \mu;$$

it is easy to see that

$$\| \mu_1 \| + \| \mu_2 \| = 1,$$

and thus

$$\mu_1{}^* = \mu_1/\| \mu_1 \|, \qquad \mu_2{}^* = \mu_2/\| \mu_2 \|$$

have the property

$$\mu = \| \mu_1 \| \mu_1{}^* + \| \mu_2 \| \mu_2{}^*,$$

which represents a contradiction, since μ is a measure representing an extreme point. The lemma is proved.

Remark. The lemma is valid also for the closure of $P(K, R)$, $\overline{P(K, R)}$ and the proof is the same.

The proof of Theorem 2.1 is now very simple. Indeed the Hahn–Banach theorem implies that any level set of $P(K, R)$ contains only one point. By the Krein–Milman and Hahn–Banach theorems combined with the Riesz–Kakutani representation theorem there exists a nonzero functional annihilating $P(K, R)$, and it is an extreme point. By the lemma, the support of the corresponding measure reduces to a point, and then our functional is of the form

$$F(f) = f(t_0),$$

for some $t_0 \in K$, which is an impossibility since no $f \in P(K, R)$ vanishes at t_0. This completes the proof of the theorem.

DEFINITION 2.3. A function $f : K \to R$ is called normal if it is continuous and $f(x) \in [0, 1]$.

The following is well known [2].

THEOREM 2.4. *For any covering* $V_1, \ldots, V_m$, *with open sets, of* K, *there exist normal functions* $h_1, \ldots, h_m$ *such that*

(1) $h_i(t) = 0$ *in the exterior of* V_i,

(2) $\sum_{i=1}^{m} h_i(t) = 1$.

We can prove the Weierstrass theorem.

THEOREM 2.5. *The space* $P(K, X)$ *is dense in* $C(K, X)^*$.

Proof. Let $f \in C(K, X)$ and $\epsilon > 0$. For each $t \in K$, let

$$V_t = \{s, \|f(t) - f(s)\| < \epsilon\},$$

which is open. Thus $\{V_t\}$ is an open covering of K and there exists a finite covering of K, say, $V_{t_1}, \ldots, V_{t_m}$. From Theorem 2.4 we find the normal functions $h_1, \ldots, h_m$ and we can construct the function

$$F^*(t) = \sum_1^m f(t_i)\, h_i(t).$$

It is easy to see that

$$\|f(t) - F^*(t)\| < \epsilon,$$

and, since each h_i can be approximated by elements of $P(K, R)$, we find easily that f can be approximated by elements of $P(K, X)$.

In a paper which is now in preparation we extend these constructions to complex Banach spaces proving in full generality a Weierstrass theorem.

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